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Calculation of Initial Vortex Roll-Up in Aircraft Wakes

John E. Yates*

Aeronautical Research Associates of Princeton, Inc., Princeton, N.J.

The initial in-plane acceleration of a vortex sheet in two-dimensional incompressible flow is calculated. The theory is used to estimate the strength and location of the discrete vortices that roll-up behind an aircraft wing. Numerical results are presented for a C-141 takeoff and a DC-9 landing configuration. The results verify a hypothesis of Donaldson[1] for estimating the strength and location of the discrete vortices that is required in a Betz-type roll-up calculation. The present theory also provides an estimate of the relative rates of vortex roll up.

Nomenclature

$A(y)$ = initial in-plane acceleration of the shed vorticity [see Eq. (7)]
A_n = Tchebichef expansion coefficients [see Eq. (12)] and Appendix
b = wingspan
$B_{nm}{}^p$ = universal constants [see Eqs. (19) and (20)]
$C_{nm}(y)$ = universal functions [see Eq. (18)]
C^p = Tchebichef expansion coefficients [see Eq. (22)]
$F(y)$ = normalized load distribution [see Eq. (4)]
$\bar{L}$ = dimensional lift/unit span
N = number of Tchebichef polynomials
q_∞ = freestream dynamic pressure
R = distance between vortex filaments [see Eq. (3)]
t = time
T_n, U_n = Tchebichef polynomial of first and second kind [see Eqs. (10) and (11)]
U_∞ = freestream velocity
V, W = velocity of vortex filament (Y,Z) along the y and z axes (see Fig. 2)
y, z = Cartesian coordinates in the wake cross plane (see Fig. 2)
(Y,Z) = Lagrange coordinates of a vortex filament
ϵ_N, ϵ_N' = relative mean square error in Tchebichef approximation of F and F', respectively, [see Eqs. (13) and (14)]
$\bar{\gamma}$ = shed vortex strength
$\bar{\Gamma}$ = bound vortex strength
$(')$ = total derivative with respect to y
$(\bar{\ })$ = denotes dimensional quantity

I. Introduction

IN a recent report by Donaldson, Snedeker, and Sullivan,[1] the general aircraft wake problem is discussed in detail. Following the ideas set down in Ref. 2, they extend the basic roll-up theory of Betz[3] to the case of flapped wings with multiple shed vortices and apply the theory to three transport aircraft in holding, takeoff, and landing configurations. The basic validity and accuracy of the extended Betz theory is verified by extensive comparison with the full-scale experiments conducted by the FAA-NAFEC (see, for example, Ref. 4.) The extended Betz theory has also been used for other load distributions by Rossow.[5]

The present paper is concerned with the first basic question of aircraft wake formation posed in Ref. 1. "What is the pattern of vorticity shed in the immediate vicinity of an aircraft and how does this vorticity pattern tend to concentrate (roll-up) behind the aircraft?" To illustrate the basic problem we refer to Fig. 1, which is a partial reproduction of Fig. 2.3 in Ref. 1. From the complex load distribution of Fig. 1, Donaldson et al., have shown with the extended Betz theory that a single rolled-up vortex is not possible. To ascertain the number and strength of shed vortices, Donaldson offers the following basic hypothesis. "The initial vorticity shed between the local minima of the curve $|d\bar{\Gamma}/d\bar{y}|$ vs $\bar{y}$ will roll-up into a discrete vortex." For the case shown in Fig. 1, the vorticity shed between $\bar{y}_A$ and the tip rolls up into a "tip" vortex, that between $\bar{y}_A$ and $\bar{y}_B$ rolls up into a "flap" vortex and the remainder rolls up into a "fuselage" vortex.

In the present paper we investigate the basic validity of the Donaldson hypothesis by calculating the initial in-plane acceleration of the shed vortex sheet. Numerical results are given for a C-141 takeoff and DC-9 landing configuration. The results generally confirm the validity of the Donaldson hypothesis and provide additional information on the relative rates of vortex roll up.

II. Formulation of the Two-Dimensional Roll-Up Problem

We consider a symmetrically loaded wing with unit semispan in a mainstream of velocity U_∞. Suppose that the shed vorticity forms a flat sheet that is initially in the plane of the wing. At time zero we release the sheet and calculate the initial roll-up velocity and acceleration in a cross plane sufficiently far downstream that the effect of the bound vortex of the wing may be neglected. It may be shown that the first-order bound vortex distortion of an initially flat vortex sheet is of order $(1/x^4)$ where x is the distance downstream in semispans. Therefore, even one

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*Senior Consultant.

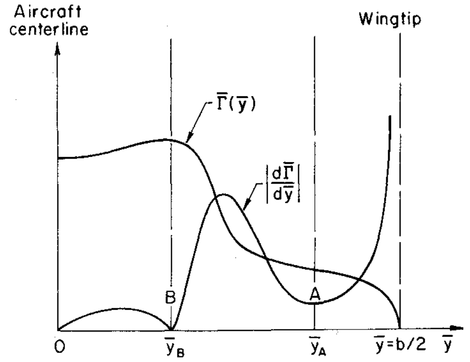

Fig. 1 Typical load distribution and its derivative for a flapped wing (see Ref. 1, Fig. 2.3).

full span downstream ($x = 2$), the bound vortex effect is small.

We consider only the two-dimensional interaction of the trailing vortex filaments and use the Biot-Savart Law to write the following pair of equations for the velocity components of a vortex filament in the y-z plane (see Fig. 2):

$$W(y,t) = -\frac{1}{\pi}\int_{-1}^{1}\frac{[Y(y,t) - Y(\eta,t)]}{R^2}F'(\eta)d\eta \quad (1)$$

$$V(y,t) = +\frac{1}{\pi}\int_{-1}^{1}\frac{[Z(y,t) - Z(\eta,t)]}{R^2}F'(\eta)d\eta \quad (2)$$

where

$$R^2 = [Y(y,t) - Y(\eta,t)]^2 + [Z(y,t) - Z(\eta,t)]^2 \quad (3)$$

and the slash indicates that the integrals are of the Cauchy principal value type. Here $Y(y,t)$ and $Z(y,t)$ are the instantaneous Cartesian coordinates of a filament that was at spanwise station y at time zero. The function $F(y)$ is the normalized section lift or bound vortex strength; i.e.,

$$F(y) = \bar{L}/q_\infty b = 2\bar{\Gamma}/U_\infty b \quad (4)$$

Therefore, the function $F'(y)$ is the negative of the shed vorticity strength.

At time zero, the vorticity is distributed along the y axis; i.e.,

$$Y(y,0) = y$$
$$Z(y,0) = 0 \quad (5)$$

With these initial conditions one could, in principle, solve Eqs. (1) and (2) and thus the complete roll-up problem.† Here we consider the short-time behavior by iterating on the initial conditions. If we substitute Eqs. (5) into Eqs. (1) and (2), we obtain at time zero

$$W(y) = \frac{1}{\pi}\int_{-1}^{1}\frac{F'(\eta)d\eta}{\eta - y}$$
$$V(y) = 0 \quad (6)$$

For convenience of notation, we let $W(y) = W(y,0)$ and $V(y) = V(y,0)$ in the following development. The first formula is recognized as the usual downwash integral in lifting line theory. The initial in-plane velocity V is zero so that the initial distortion of the vortex sheet is normal to the sheet and proportional to W. To determine the initial motion in the plane of the sheet, we evaluate the time derivative of V at time zero. From Eq. (2) we get

†The continuous formulation of the roll-up problem may be a practical alternative to the discrete vortex model considered by Westwater[6] and others.[7]

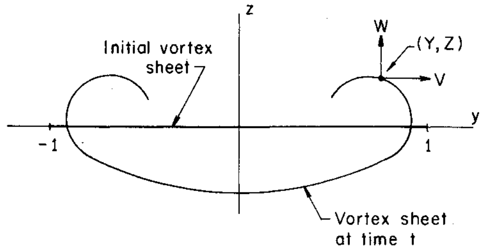

Fig. 2 Vortex roll-up in a typical wake cross plane.

$$A(y) = \dot{V}(y,0) = -\frac{1}{\pi}\int_{-1}^{1}\left[\frac{W(y) - W(\eta)}{y - \eta}\right]\frac{F'(\eta)d\eta}{\eta - y} \quad (7)$$

Vortex filaments initially move outboard or inboard as the acceleration function $A(y)$ is positive or negative. At a point where $A(y) = 0$ and $A'(y) > 0$, the vorticity initially separates. We shall interpret these zeroes of $A(y)$ as the points for dividing up the vorticity in a Betz roll-up calculation, as presented in Ref. 1. The points where $A(y) = 0$ and $A'(y) < 0$ are interpreted as centers of roll up. The magnitude of $A(y)$ gives a measure of the rate of roll up into discrete vortices.

III. Evaluation of the Initial Roll-Up Acceleration for a Symmetrically Loaded Wing

Consider a symmetrically loaded wing with an elliptically loaded tip. The shed vorticity distribution can be expressed conveniently as a series of Tchebichef polynomials of the first kind; i.e.,

$$F_N'(y) = \frac{2}{(1 - y^2)^{1/2}}\sum_{n=1,3,5}^{N} A_n T_n(y) \quad (8)$$

The actual load distribution is a Tchebichef series of the first kind,

$$F_N(y) = -2(1 - y^2)^{1/2}\sum_{n=1,3,5}^{N}\frac{A_n}{n}U_{n-1}(y) \quad (9)$$

Both Tchebichef polynomials satisfy the recursion formula.[8]

$$Q_{n+1} = 2yQ_n - Q_{n-1} \quad (10)$$

with the starting values

$$Q_0 = 1,\ Q_1 = y \quad \text{when} \quad Q_n = T_n$$
$$Q_0 = 1,\ Q_1 = 2y \quad \text{when} \quad Q_n = U_n \quad (11)$$

The coefficients A_n are defined by a single quadrature over the load distribution

$$A_n = -\frac{2n}{\pi}\int_0^1 U_{n-1}(y)F(y)dy \quad (12)$$

and must be evaluated numerically. A simple algorithm is given in the Appendix. The relative mean-square errors in the Tchebichef approximations of F and F' are given by the following formulas:

$$\epsilon_N = 1 - \pi\sum_{n=1,3,5}^{N} A_n^2/n^2\int_0^{\pi/2}F^2(\cos\theta)d\theta \quad (13)$$

and

$$\epsilon_N' = 1 - \pi\sum_{n=1,3,5}^{N} A_n^2/\int_0^{\pi/2}F'^2(\cos\theta)\sin^2\theta d\theta \quad (14)$$

The accuracy required in the approximation of F' determines the number N of Tchebichef polynomials that must be used in the calculation. To evaluate the downwash

$W(y)$ we substitute the Tchebichef approximation of $F'(y)$ [see Eq. (8)] into the first of Eqs. (6). We get

$$W(y) = 2 \sum_{n=1,3,5}^{N} A_n \cdot \frac{1}{\pi} \int_{-1}^{1} \frac{T_n(\eta) d\eta}{(\eta - y)(1-\eta^2)^{1/2}}$$

$$= 2 \sum_{n=1,3,5}^{N} A_n U_{n-1}(y) \qquad (15)$$

The integral relations for Tchebichef polynomials needed to obtain these results may be found in Ref. 8. Next, we substitute Eqs. (8) and (15) into Eq. (7) to get

$$A(y) = -4 \sum_{n=1,3,5}^{N} \sum_{m=1,3,5}^{N} A_n A_m \cdot C_{nm}(y) \qquad (16)$$

where

$$C_{nm}(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{U_{n-1}(y) - U_{n-1}(\eta)}{y - \eta} \cdot \frac{T_m(\eta) d\eta}{(\eta - y)(1-\eta^2)^{1/2}} \qquad (17)$$

The functions $C_{nm}(y)$ are universal functions that do not depend on the load distribution. Also, we note that they are polynomials of degree $n + m - 3$ and may be re-expressed as a Tchebichef series of odd degree only; i.e.,

$$C_{nm}(y) = \sum_{p=1,3,\ldots}^{n+m-3} B_{nm}{}^{p} U_p(y) \qquad (18)$$

where

$$B_{nm}{}^{p} = \frac{2}{\pi} \int_{-1}^{1} (1-y^2)^{1/2} U_p(y) C_{nm}(y) dy \qquad (19)$$

The coefficients $B_{nm}{}^{p}$ are pure integers and are easily evaluated with the following formulas‡:

For

$$p \geq 1,\ m \geq 1,\ n \geq 3 \text{ and } p, m, n \text{ odd}$$

$$B_{nm}{}^{p} = 0 \text{ when } p \geq n + m - 1 \text{ or } p \leq m - n - 1$$

$$B_{nm}{}^{p} = 2(p+1)(n-p-1) \quad l \leq m \leq n - p - 1$$

$$B_{nm}{}^{p} = \frac{(p - m + n + 1)(m - p + n - 1)}{2} \quad m \geq n - p + 1 \qquad (20)$$

With Eq. (18) we can express $A(y)$ in the convenient form

$$A(y) = 4 \sum_{p=1,3,\ldots}^{2N-3} C^p U_p(y) \qquad (21)$$

where

$$C^p = -\sum_{n=3,5}^{N} \sum_{m=1,3,5}^{N} B_{nm}{}^{p} A_n A_m \qquad (22)$$

Finally, we remark that the dimensional load $\bar{L}$, vorticity $\bar{\gamma}$, downwash velocity $\bar{W}$, and in-plane acceleration $\bar{A}$, may be calculated with the following formulas:

$$\bar{L} = q_\infty b F(y)$$

$$\bar{\gamma} = -U_\infty F'(y) \quad \text{shed vorticity/unit span}$$

$$\bar{W} = U_\infty W(y) \quad \text{downwash velocity}$$

$$\bar{A} = \frac{2U_\infty^2}{b} \cdot A(y) \quad \text{in-plane acceleration} \qquad (23)$$

IV. Numerical Calculations

We consider two example aircraft that were treated by Donaldson et al. in Ref. 1. Each aircraft has a "T-tail" so that any effect of the horizontal tail vortex system on the roll-up process of the wing is expected to be minimal compared to an aircraft with a conventional horizontal tail. The first is a C-141 in a takeoff configuration with 31° flap angle. The load, downwash, vorticity, and acceleration functions are plotted in Fig. 3a and b. The roll-up centers for the tip vortex, outboard flap vortex, and inboard flap (fuselage) vortex are well defined by the acceleration function and are, respectively, at the stations 0.96, 0.66, and 0.09 of the semispan (see points A′, B′, C of Fig. 3b). Also, we calculate a weaker roll-up center at station 0.278 that is approximately the inboard engine location which is at station 0.3. The roll-up centers are in perfect agreement with the Donaldson hypothesis. The points where the vorticity separates are not as sharply defined by the Donaldson hypothesis although they are close enough for engineering calculations with the Betz theory.

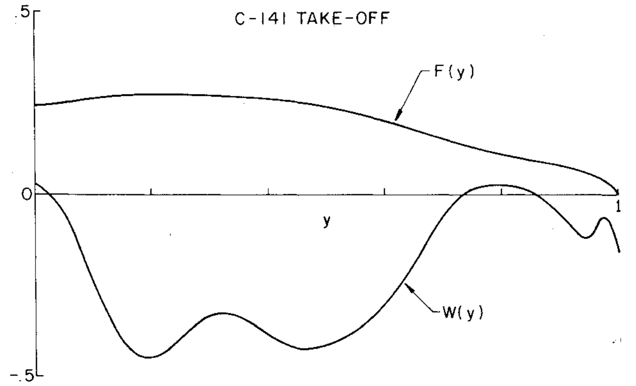

Fig. 3a Load and downwash distribution for a C-141 takeoff configuration; flap angle = 31°.

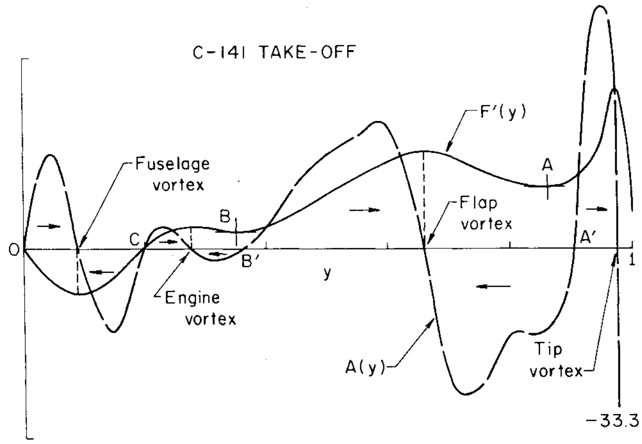

Fig. 3b Shed vorticity and acceleration function for a C-141 takeoff configuration; flap angle = 31°.

The magnitude of the acceleration function gives some information on the relative rates of roll up of the various vortices. For example, the tip vortex acceleration is considerably greater than that of either flap vortex. The engine vortex acceleration is much slower than the flap vortices and could possibly be absorbed by the flap vortices before it rolls up about its own center. The ultimate fate of the shed vorticity can only be determined by more detailed roll-up calculations. We remark finally that 13 Tchebichef polynomials were required to approximate the shed vortex distribution to approximately 1% accuracy [$\epsilon_N' = 0.012$, see Eq. (14)].

Our second example is a DC-9 in a landing configuration with 50 ° flap angle. The results are presented in Fig. 4. The flap vortex center is well defined at station 0.632 and agrees with the Donaldson hypothesis. The tip vortex

‡The author is indebted to R. Sullivan for his assistance in evaluating these coefficients.

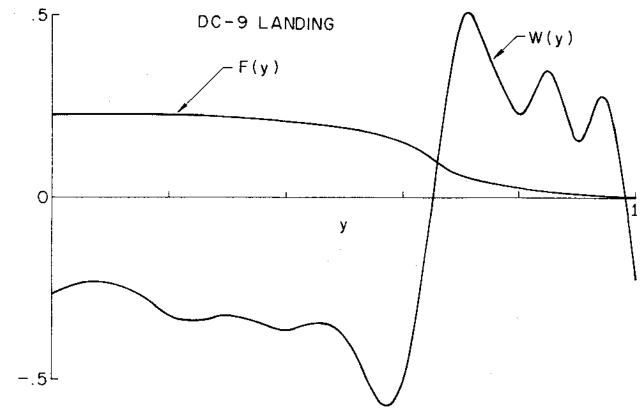

Fig. 4a Load and downwash distribution for a DC-9 landing configuration; flap angle = 50°.

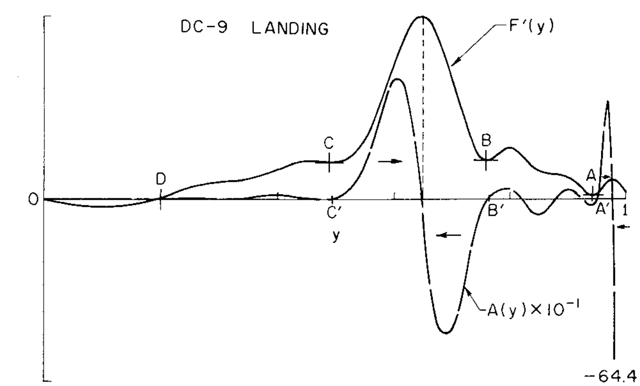

Fig. 4b Shed vorticity and acceleration function for a DC-9 landing configuration; flap angle = 50°.

is much weaker for the present case than the C-141 and is not as clearly defined by the acceleration functions or the Donaldson hypothesis. We note that the tip acceleration rate is very large compared to that of the flap vortex. Since the tip vortex is very weak compared to the flap vortex, it is of little interest for the purpose of doing a Betz roll-up calculation. However, the relatively large and spurious values of the tip acceleration indicate that much care must be exercised when detailed roll-up calculations are performed.

For the DC-9 calculation, 15 Tchebichef polynomials were required to obtain a 7% accuracy ($\epsilon_N' = 0.07$). The majority of the error is near the tip which is not elliptically loaded in the present case (Fig. 4a).

V. Conclusions

Based on results of the present study, we conclude that the Donaldson hypothesis is satisfactory for predicting the location and strength of the discrete vortices for use in the Betz theory. However, the short-time roll-up theory presented in the present paper gives a more rational estimate that is very simple to carry out numerically. Also, it provides some additional information on the relative rates of roll-up. It is recommended that the initial roll-up calculation be made as preliminary to use of the extended Betz theory.

Appendix

To evaluate the Tchebichef coefficients A_n given by Eq. (12) in the paper, we introduce trigonometric variables, and use a piecewise linear approximation of F. We get

$$A_n = \frac{2}{\pi} \sum_{k=2}^{N} w_k{}^n \cdot F_k \quad n = 1,3,5,\ldots \tag{A1}$$

where

$$w_k{}^n = -\frac{2}{n}\left[\frac{\sin^2(nh_k/2)}{h_k} + \frac{\sin^2(nh_{k-1}/2)}{h_{k-1}}\right]\sin n\theta_k + \frac{1}{n}\left[\frac{\sin nh_k}{h_k} - \frac{\sin nh_k}{h_{k-1}}\right]\cos n\theta_k$$

$$k = 2,\ldots, N-1$$

$$w_N{}^n = -\frac{2\sin^2(nh_{N-1}/2)}{nh_{N-1}} \sin\frac{n\pi}{2} \tag{A2}$$

$$\theta_1 = 0, \quad \theta_N = \pi/2$$

$$h_k = \theta_{k+1} - \theta_k, \quad k = 1,2,\ldots, N-1$$

$$y_k = \cos\theta_k$$

$$F_k = F(y_k) \tag{A3}$$

When equal-spaced θ points are used, the second term in the weighting coefficient (see the first of Eqs. (A2) is zero).

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